Low-temperature expansion in link formulation. II

O. Borisenko[1], **V. Kushnir**[2]

N.N.Bogolyubov Institute for Theoretical Physics, National Academy of Sciences of Ukraine, 252143 Kiev, Ukraine

Abstract

We extend our previous analysis to arbitrary two dimensional $SU(N)$ principal chiral model in a link formulation. A general expression for the second order coefficient of fixed distance correlation function is given in terms of Green functions. This coefficient is calculated for distance 1 and is proven to be path independent. We also study the weak coupling expansion of the free energy of one dimensional $SU(N)$ model and explain why it is non-uniform in the volume. Further, we investigate the contribution of holonomy operators to the low-temperature expansion in two dimensions. All our results agree with the conventional expansion. Nevertheless, we give some example which indicates that the expansion in the link formulation can also suffer from ambiguities previously found in the conventional perturbation theory.

1 Introduction

In our previous paper [1] we proposed to use an invariant link formulation to investigate some properties of two dimensional ($2D$) models in the low temperature limit. We have argued that this approach is more reliable for calculation of asymptotic expansions of invariant functions in cases when the Mermin-Wagner theorem forbids spontaneous symmetry breaking in the thermodynamic limit. We have calculated the first order coefficient of the free energy of $SU(2)$ model which turned out to coincide with the conventional answer. The starting point of calculations is the following partition function of $2D$ $SU(N)$ model

$$Z = \int \prod_l dV_l \exp\left[\beta \sum_l \mathrm{Re}\ \mathrm{Tr} V_l + \ln J(V)\right] , \tag{1}$$

where the Jacobian $J(V)$ is a product of $SU(N)$ delta-functions taken over all plaquettes of $2D$ lattice

$$J(V) = \prod_p \left[\sum_r d_r \chi_r \left(\prod_{l\in p} V_l\right)\right] . \tag{2}$$

[1]email: oleg@ap3.bitp.kiev.ua
[2]email: kushnir@ap3.bitp.kiev.ua

The sum over r is sum over all representations of $SU(N)$, $d_r = \chi_r(I)$ is the dimension of r-th representation. The $SU(N)$ character χ_r depends on a product of the link matrices V_l along a closed path[3]:

$$\prod_{l\in p} V_l = V_n(x)V_m(x+n)V_n^+(x+m)V_m^+(x) \ . \tag{3}$$

One of the main technical observations made in [1] is that at large β the $SU(N)$ delta-function can actually be replaced by the Dirac delta-function so that partition function becomes

$$Z(\beta >> 1) = \int \prod_l dV_l \exp\left[\beta \sum_l \text{Re Tr}V_l\right] \prod_{p,k} \delta(\omega_p^k) \ , \tag{4}$$

where ω_p^k is a plaquette angle

$$\prod_{l\in p} V_l = \exp(i\lambda^k \omega_p^k) \ . \tag{5}$$

Such a naive replacement is of course plagued by the problem of (N^2-1) zero modes for auxiliary fields therefore the delta-function in (4) should be correspondingly regularized. This can be done, for instance, by introducing the heat-kernel into the expression for the partition function instead of the $SU(N)$ delta-function. This procedure is equivalent to introducing a mass term for auxiliary fields. Direct calculations show however that zero modes decouple from the large-β expansion in the $SU(2)$ model [1] (we could not generalize that consideration for arbitrary $SU(N)$, though). In what follows we work with massless Green functions omitting zero modes from all lattice sums similarly to $SU(2)$ case. All general expressions remain valid if one works with mass regulator term as mentioned above. Since all logarithmic divergences cancel we expect that the convergence to the thermodynamic limit (TL) is uniform: in all cases the final result can be expressed in terms of link functions $G_{ll'}$, $D_l(x')$ introduced in [1] and standard D-functions. For $SU(N)$ model the expansion itself is done precisely like in [1] for $SU(2)$. We obtain for the first order coefficient of the free energy

$$\frac{1}{2L^2} C^1_{f.e.} = \frac{(N^2-1)(N^2-2)}{64N} \ . \tag{6}$$

This result agrees with the result of the conventional PT [2].

An essential motivation of our investigations is to study the problem of the uniformity of the low-temperature expansion in $2D$ models, the question addressed in [3]. Before direct attack of this problem we have decided we need to make our approach more suitable for practical computations. Also, it is necessary to establish some general properties of the expansion, for example the path independence of the correlation functions and the infrared finitness of the expansion. We address these and some other problems in the present article. We give a general expression for the second order coefficient of fixed distance correlation function in terms of Green functions and calculate this coefficient for distance 1. Our final result agrees with Eq.(6).

[3]We take plaquettes for simplicity though one could choose any set of L^2 independent loops.

Let us consider two-point correlation function

$$\Gamma(x,y) =< \frac{1}{N}\mathrm{Re\ Tr}\ U_x U_y^+ >=< \frac{1}{N}\mathrm{Re\ Tr}\prod_{l\in C_{xy}} W_l > , \tag{7}$$

where $W_l = V_l$ if along the path C_{xy} the link l goes in the positive direction and $W_l = V_l^+$, otherwise. Obviously, it does not depend on a shape of the path C_{xy} and any expansion in the link formulation must respect this property. In large-β expansion every coefficient at a given order should be path independent. This requirement generates certain relations on link Green functions thus allowing for a deformation of any given path to any other one. Since this independence is guaranteed only by the Jacobian $J(V)$, the check on such independence would be highly desirable since it can show whether we treat the Jacobian correctly and whether all contributions are taken into account.

Further, we re-analyze the free energy of $1D$ models. As is well known, large-β expansion in $1D$ non-abelain models is non-uniform in the volume, therefore our expansion has to explain this feature.

As for now, all our results agree completely with the standard approach to the low-temperature expansion in $2D$ models. Nevertheless, we find that the expansion in the link formulation also suffers from ambiguities previously found in the standard expansion [3]. Again, problem arises only for non-abelian models and only starting from $1/\beta^2$. We mention this example in the Discussion.

This paper is organized as follows. In the next section we describe some properties of link Green functions. In Section 3 we study the correlation function in $2D$ models. We calculate the fixed distance correlation function in the XY model. The general expansion of the second order coefficient of the correlation function in $SU(N)$ model is given in terms of Green functions. In this paper we compute this coefficient for distance 1. A proof of the path independence of the first and second order coefficients of the correlation function is given in Section 4. Also, we prove here the infrared finitness of the second order coefficient. Section 5 is devoted to one dimensional models. We analyze the low-temperature expansion of the free energy and explain why this expansion is non-uniform in the volume. In Section 6 we study contribution of the holonomy operators to the low-temperature expansion in $2D$ models and compare it with one dimensional case. We summarize our results in Section 7.

2 Link Green functions

The main building blocks of the low-temperature expansion in the link formulation are link Green functions $G_{ll'}$ and $D_l(x')$ introduced in [1]. In this section we describe some of their basic properties. The functions $G_{ll'}$ and $D_l(x')$ are defined as

$$G_{ll'} = 2\delta_{l,l'} - G_{x,x'} - G_{x+n,x'+n'} + G_{x,x'+n'} + G_{x+n,x'} , \tag{8}$$

$$D_l(x') = G_{x,x'} - G_{x+n,x'} , \tag{9}$$

where link $l = (x, n)$ is defined by a point x and a positive direction n. $G_{x,x'}$ is a "standard" Green function on the periodic lattice

$$G_{x,x'} = \frac{1}{L^2} \sum_{k_n=0}^{L-1} \frac{e^{\frac{2\pi i}{L} k_n (x-x')_n}}{f(k)} \ , \ k_n^2 \neq 0 \ , \tag{10}$$

where we have denoted

$$f(k) = 2 - \sum_{n=1}^{2} \cos \frac{2\pi}{L} k_n \ . \tag{11}$$

Normalization is such that $G_{ll} = 1$. In the momentum space $G_{ll'}$ reads

$$\begin{aligned} G_{ll'} = \frac{2\delta_{nn'} - 1}{L^2} \sum_{k_n=0}^{L-1} \frac{e^{\frac{2\pi i}{L} k_n (x-x')_n}}{f(k)} F(n, n') + \frac{2\delta_{nn'}}{L^2} \ , \\ F(n = n') = 2(1 - \cos \frac{2\pi}{L} k_p) \ , n \neq p \ , \\ F(n \neq n') = (1 - e^{\frac{2\pi i}{L} k_n})(1 - e^{-\frac{2\pi i}{L} k_{n'}}) \ . \end{aligned} \tag{12}$$

Using this representation it is easy to prove the following "orthogonality" relations for the link functions

$$\sum_b G_{lb} G_{bl'} = 2 G_{ll'} \ , \tag{13}$$

$$\sum_b D_b(x) G_{bl'} = 0 \ , \tag{14}$$

$$\sum_b D_b(x) D_b(x') = 2 G_{x,x'} \ , \tag{15}$$

where the sum over b runs over all links of $2D$ lattice. Let C_{xy} be some path connecting points x and y and let C_{xy}^d be a path dual to the path C_{xy}, i.e consisting of the dual links which are orthogonal to the original links $l, l' \in C_{xy}$. For simplicity let us consider a path consisting of links which point only in a positive direction (see Fig.1 for our notation of links on a dual lattice). We then have

$$\sum_{l,l' \in C_{xy}^d} G_{ll'} = 2D(x-y) \ , \tag{16}$$

where

$$D(x) = \frac{1}{L^2} \sum_{k_n=0}^{L-1} \frac{1 - e^{\frac{2\pi i}{L} k_n x_n}}{f(k)} \ , \ k_n^2 \neq 0. \tag{17}$$

Let $\mathcal{L}$ be any closed path. Then

$$\sum_{l,l' \in \mathcal{L}} \bar{G}_{ll'} = 0 \ , \tag{18}$$

where $\bar{G}_{ll'} = G_{ll'}$ if both link l and l' point in either positive or negative direction and $\bar{G}_{ll'} = -G_{ll'}$ if one (and only one) of links points in negative direction. Let us

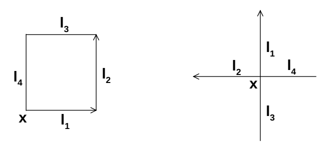

Figure 1: Plaquette of original lattice and links of dual lattice. Link is determined by point x and a positive direction, e.g. $l_3 = (x - n_1; n_1)$.

demonstrate this property for the simplest closed path, i.e. for a plaquette. From Fig.1 and definition (8) we have on the dual lattice[4]

$$G_{l_1 l_2} = G_{l_3 l_4} = -G_{l_1 l_4} = -G_{l_2 l_3} = -1 + \frac{2}{\pi} \ \ , G_{l_1 l_3} = G_{l_2 l_4} = -1 + \frac{4}{\pi} \ . \tag{19}$$

We obtain from (18) in case of plaquette

$$\sum_{l,l' \in p} \bar{G}_{ll'} = (4 + 8G_{l_1 l_2} - 4G_{l_1 l_3}) = 0 \ . \tag{20}$$

One sees that $G_{ll'}$ satisfies the following equation

$$G_{l_1 l'} + G_{l_2 l'} - G_{l_3 l'} - G_{l_4 l'} = 0 \tag{21}$$

for any link l'. $D_l(x')$ satisfies the lattice Laplas equation

$$D_{l_1}(x') + D_{l_2}(x') - D_{l_3}(x') - D_{l_4}(x') = 2\delta_{x,x'} \ . \tag{22}$$

We shall also use the following equality

$$\sum_x (G_{l_1 b_1} G_{l_2 b_2} + G_{l_3 b_1} G_{l_4 b_2} - G_{l_1 b_2} G_{l_2 b_1} - G_{l_3 b_2} G_{l_4 b_1}) = 0 \ , \tag{23}$$

which is valid for any links b_1, b_2[5].

3 Correlation function in two-dimensional models

Throughout this section we work on a dual lattice and follow notations of the previous section. Links b_i denote links belonging to the path C^d_{xy}.

[4] Numerical values for G_{l_i,l_j} refer to the thermodynamic limit.

[5] On a finite lattice equations (13)-(15) and (21)-(23) are valid only up to $O(1/L^2)$ corrections.

3.1 Correlation function in the XY model

In our previous paper we considered low-temperature expansion of the free energy of the XY model. Here we want to demonstrate how conventional results for the fixed distance correlation function can be recovered in the link formulation. We refer to [1] for expansion of the Gibbs measure and calculation of the generating functional in the link formulation. For the fixed-distance correlation function we get

$$\Gamma_{XY}(x,y) = < \prod_{l \in C^d_{xy}} e^{i\phi_l} > = 1 - \frac{1}{\beta}\Gamma^{(1)}(x,y) + \frac{1}{\beta^2}\Gamma^{(2)}(x,y) + O(\beta^{-3}) \ . \tag{24}$$

Coefficients $\Gamma^{(i)}$ are given by

$$\Gamma^{(1)}(x,y) = \frac{1}{4}\sum_{b_1 b_2} G_{b_1 b_2} \ , \tag{25}$$

$$\Gamma^{(2)}(x,y) = \frac{1}{32}\left(\sum_{b_1 b_2} G_{b_1 b_2}\right)^2 - \frac{1}{32}\sum_{b_1 b_2}\sum_{l} G_{b_1 l} G_{l b_2} \ . \tag{26}$$

Using orthogonality relation (13) and Eq.(16) we obtain

$$\Gamma_{XY}(x,y) = 1 - \frac{1}{2\beta}D(x-y) + \frac{1}{8\beta^2}D(x-y)\left[D(x-y) - 1\right] + O(\beta^{-3}) \ . \tag{27}$$

This result coincides with that quoted in the literature for $O(2)$ model [4].

3.2 Correlation function in $SU(N)$ model: general expansion

In this subsection we give general expression for the second-order coefficient of the fixed-distance correlation function in terms of Green functions. The generating functional is diagonal in group indices and coincides with that given in [1] for $SU(2)$. Again, the expansion of the Gibbs measure and the Jacobian is done precisely like in [1]. Expanding (7) we write down

$$\Gamma_{SU(N)}(x,y) = 1 - \frac{1}{\beta}\Gamma^{(1)}(x,y) + \frac{1}{\beta^2}\Gamma^{(2)}(x,y) + \ldots \ . \tag{28}$$

The first coefficient is given by

$$\Gamma^{(1)}(x,y) = \frac{N^2-1}{4N}\sum_{b_1 b_2} G_{b_1 b_2} = \frac{N^2-1}{2N}D(x-y) \tag{29}$$

and coincides with the conventional result. We split the second coefficient into three pieces

$$\Gamma^{(2)}(x,y) = \frac{N^2-1}{16}\left(Q_1 + Q_2 + Q_3\right) \ . \tag{30}$$

Q_1 describes contribution from the second order term of the correlation function and zero order term of the Gibbs measure

$$Q_1 = \frac{1}{N^2} \sum_{k=1}^{4} Q_1^{(k)} \ , \tag{31}$$

where

$$Q_1^{(1)} = \frac{2N^2-3}{6} \sum_b G_{bb}^2 \ , \tag{32}$$

$$Q_1^{(2)} = \sum_{b_1>b_2} \Big[(N^2-1)G_{b_1b_1} + (N^2-2)G_{b_1b_2}^2 + \frac{4}{3}(2N^2-3)G_{b_1b_2}\Big] \ , \tag{33}$$

$$Q_1^{(3)} = \sum_{b_1\neq b_2\neq b_3} \Big[(N^2-1)G_{b_1b_2} + (N^2-2)G_{b_1b_2}G_{b_2b_3}\Big] \ , \tag{34}$$

$$Q_1^{(4)} = 4 \sum_{b_1>b_2>b_3>b_4} \Big[(N^2-1)(G_{b_1b_2}G_{b_3b_4} + G_{b_1b_4}G_{b_2b_3}) - G_{b_1b_3}G_{b_2b_4}\Big] \ . \tag{35}$$

Q_2 describes contribution of $\beta^{-3/2}$ order from the expansion of correlation function and of $\beta^{-1/2}$ order from the expansion of the Jacobian. This "self-connected" piece is given by

$$\begin{aligned} Q_2 = -\sum_{b_1>b_2>b_3} \sum_x \sum_{i<j}^{4} [&Q_x^{ij}(b_1,b_2,b_3) - Q_x^{ij}(b_1,b_3,b_2) + \\ Q_x^{ij}(b_2,b_3,b_1) - Q_x^{ij}(b_2,b_1,b_3) &+ Q_x^{ij}(b_3,b_1,b_2) - Q_x^{ij}(b_3,b_2,b_1)] \ , \end{aligned} \tag{36}$$

where

$$Q_x^{ij}(b_1,b_2,b_3) = D_{b_1}(x)G_{b_2l_i}G_{b_3l_j} \ . \tag{37}$$

Finally, there are contributions of the first order terms from the expansion of correlation function, the Gibbs measure and the Jacobian. Q_3 describes the corresponding connected pieces

$$Q_3 = \sum_{k=1}^{3} Q_3^{(k)} \ , \tag{38}$$

where

$$Q_3^{(1)} = \frac{1}{2N^2} \sum_{b_1b_2} \sum_l G_{b_1l}G_{lb_2} \ , \tag{39}$$

$$\begin{aligned} Q_3^{(2)} = -\frac{2}{3} \sum_{b_1b_2} \sum_x [&3D_{b_2}(x)(G_{b_1l_1} - G_{b_1l_4})(G_{l_1l_2} - G_{l_1l_3}) + \\ &\frac{1}{2}\sum_{i=1}^{4} G_{b_1l_i}G_{b_2l_i}(D_{l_1}(x) + D_{l_2}(x) - D_{l_3}(x) - D_{l_4}(x)) + \\ (G_{b_1l_1} + G_{b_1l_2})(2D_{l_1}(x)G_{b_2l_2} &- 2D_{l_4}(x)G_{b_2l_3} - D_{l_2}(x)G_{b_2l_1} + D_{l_3}(x)G_{b_2l_4}) + \\ (D_{l_1}(x) + D_{l_2}(x))G_{b_1l_3}G_{b_2l_4} &- (D_{l_3}(x) + D_{l_4}(x))G_{b_1l_1}G_{b_2l_2}] \ , \end{aligned} \tag{40}$$

$$Q_3^{(3)} = \frac{1}{4} \sum_{b_1 b_2} \sum_{x,x'} \sum_{i<j}^{4} \sum_{i'<j'}^{4} \left(G_{x,x'} I_1 + I_2 \right) \ . \qquad (41)$$

We have denoted

$$I_1 = G_{b_1,l_i} G_{b_2,l'_{i'}} G_{l_j,l'_{j'}} + G_{b_1,l_j} G_{b_2,l'_{j'}} G_{l_i,l'_{i'}} - \\ G_{b_1,l_i} G_{b_2,l'_{j'}} G_{l_j,l'_{i'}} - G_{b_1,l_j} G_{b_2,l'_{i'}} G_{l_i,l'_{j'}} \ , \qquad (42)$$

$$I_2 = D_{b_1}(x) D_{b_2}(x') (G_{l_i,l'_{j'}} G_{l_j,l'_{i'}} - G_{l_i,l'_{i'}} G_{l_j,l'_{j'}}) + \qquad (43) \\ 2 D_{b_1}(x) D_{l_i}(x') (G_{b_2,l'_{i'}} G_{l_j,l'_{j'}} - G_{b_2,l'_{j'}} G_{l_j,l'_{i'}}) + \\ 2 D_{b_1}(x) D_{l_j}(x') (G_{b_2,l'_{j'}} G_{l_i,l'_{i'}} - G_{b_2,l'_{i'}} G_{l_i,l'_{j'}}) + \\ D_{l'_{i'}}(x) G_{b_1,l'_{j'}} (D_{l_i}(x') G_{b_2,l_j} - D_{l_j}(x') G_{b_2,l_i}) + \\ D_{l'_{j'}}(x) G_{b_1,l'_{i'}} (D_{l_j}(x') G_{b_2,l_i} - D_{l_i}(x') G_{b_2,l_j}) \ .$$

In all formulae link l_i ($l'_{j'}$) refers to one of four links attached to a given site x (x') (see Fig.1). As it stays, this expression for the second order coefficient is valid for any path if all links $b_i \in C^d_{xy}$ point in positive directions. If one considers a path where some links point in a negative direction, one has to change a sign of the corresponding Green functions.

3.3 $\Gamma_{SU(N)}(1,0)$

Here we compute correlation function for distance 1. From (28)-(30) we have

$$\Gamma_{SU(N)}(1,0) = 1 - \frac{1}{\beta} \frac{N^2-1}{4N} + \frac{1}{\beta^2} \frac{N^2-1}{16} \left(Q_1 + Q_3 \right) \ , \qquad (44)$$

where we took into account that $D(1,0) = 1/2$ and $Q_2 = 0$ in this case. Since only $Q_1^{(1)}$ contributes to Q_1 we easily find

$$Q_1 = \frac{2N^2-3}{6N^2} \ . \qquad (45)$$

To calculate Q_3 we first note that one has to put $b_1 = b_2 = b$, where b is any fixed link. From (39) we obtain using (13)

$$Q_3^{(1)} = \frac{1}{N^2} \ . \qquad (46)$$

To calculate $Q_3^{(2)}$ we make use the translation invariance to write down

$$Q_3^{(2)} = \frac{1}{2L^2} \sum_b Q_3^{(2)} \ . \qquad (47)$$

Using again (13)-(14) we find

$$Q_3^{(2)} = -\frac{2}{3}\left(3 + 2G_{l_1 l_3} - G_{l_1 l_2}\right) = -\frac{4}{3}\left(1 + \frac{3}{\pi}\right) \ . \tag{48}$$

The same tricks applied to $Q_3^{(3)}$ give

$$Q_3^{(3)} = \frac{1}{2}\left(B_1 + B_2\right) \ , \tag{49}$$

where

$$B_1 = \frac{1}{2L^2}\sum_{x,x'}\sum_{i<j}^{4}\sum_{i'<j'}^{4} G_{x,x'}(G_{l_i,l'_{i'}}G_{l_j,l'_{j'}} - G_{l_i,l'_{j'}}G_{l_j,l'_{i'}}) \ , \tag{50}$$

$$B_2 = \frac{1}{2L^2}\sum_{x,x'}\sum_{i<j}^{4}\sum_{i'<j'}^{4}(G_{l_i,l'_{i'}}D_{l'_{j'}}(x)D_{l_j}(x') + G_{l_j,l'_{j'}}D_{l'_{i'}}(x)D_{l_i}(x') - \\ G_{l_i,l'_{j'}}D_{l'_{i'}}(x)D_{l_j}(x') - G_{l_j,l'_{i'}}D_{l'_{j'}}(x)D_{l_i}(x')) \ . \tag{51}$$

B_1 and B_2 have been calculated in [1] and equal

$$B_1 = \frac{1}{2} \ , \quad B_2 = \frac{8}{\pi} + 1 \ . \tag{52}$$

We thus find

$$Q_1 + Q_3 = -\frac{N^2 - 2}{4N^2} \ . \tag{53}$$

It leads to a final result

$$\Gamma_{SU(N)}(1,0) = 1 - \frac{1}{\beta}\frac{N^2-1}{4N} - \frac{1}{\beta^2}\frac{(N^2-1)(N^2-2)}{64N^2} \ , \tag{54}$$

which agrees with result for the free energy, Eq.(6).

4 Path independence of $\Gamma_{SU(N)}(x,y)$

We want to prove now that our representation for the correlation function is path independent. In the strong coupling region such an independence is a simple consequence of the compact invariant integration which is lost in the large-β expansion. Since, however any coefficient at a given order of the large-β expansion has to be path independent the direct proof of this property serves as an additional check of the correctness of the expansion. The path independence is guaranteed by the Jacobian which introduces the constraint on the plaquette matrix $V_p = 1$. Just this constraint imposes independence of the correlation function of the path and this constraint must be checked at each order of the large-β expansion. To show that this constraint is fulfilled we write down

$$\Gamma_p = <\frac{1}{N}\mathrm{Re}\ \mathrm{Tr}V_p> = 1 - \frac{1}{\beta}\Gamma_p^{(1)} + \frac{1}{\beta^2}\Gamma_p^{(2)} \ . \tag{55}$$

The path independence requires

$$\Gamma_p^{(1)} = \Gamma_p^{(2)} = 0 \ .$$

First, we note that the expansion for the correlation function given in section 3.2 remains valid also for Γ_p where one has to identify links b_i with links belonging to a given plaquette as shown in Fig.1. Besides, one has to change a sign in some Green functions. If we write

$$V_p = V_{l_1} V_{l_2} V_{l_3}^+ V_{l_4}^+$$

the signs should be changed in G_{ll_3}, G_{ll_4}, $D_{l_3}(x)$ and $D_{l_4}(x)$.

The equality $\Gamma_p^{(1)} = 0$ trivially follows from (18). We parametrize $\Gamma_p^{(2)}$ as

$$\Gamma_p^{(2)} = \frac{N^2-1}{16}\left(\frac{1}{N^2}C_1 + C_2\right) \ . \tag{56}$$

Since N is arbitrary, the coefficients C_1 and C_2 should be equal to zero independently. There are two contributions to C_1 coming from Q_1 and from $Q_3^{(1)}$. We find via direct computation

$$\begin{aligned} C_1 = -\frac{1}{2}\sum_b G_{bb}^2 - \sum_{b_1>b_2}\left[1 + 2G_{b_1b_2}^2 + 4G_{b_1b_2}\right] - \\ 2\sum_{b_1>b_2>b_3}\left[G_{b_1b_2} + G_{b_2b_3} + G_{b_1b_3} + 2(G_{b_1b_2}G_{b_2b_3} + G_{b_1b_3}G_{b_3b_2} + G_{b_1b_2}G_{b_1b_3})\right] - \\ 4\sum_{b_1>b_2>b_3>b_4}\left[G_{b_1b_2}G_{b_3b_4} + G_{b_1b_4}G_{b_2b_3} + G_{b_1b_3}G_{b_2b_4}\right] + \\ \frac{1}{2}\sum_{b_1b_2}\sum_l G_{b_1l}G_{lb_2} = 0. \end{aligned} \tag{57}$$

C_2 consists of the following terms

$$C_2 = Q_1(p) + Q_2(p) + Q_3^{(2)}(p) + Q_3^{(3)}(p) \ , \tag{58}$$

where we have introduced obvious definitions. For $Q_1(p)$ (terms at N^2 in (32)-(35)) we get

$$\begin{aligned} Q_1(p) = \frac{1}{3}\sum_b G_{bb}^2 + \sum_{b_1>b_2}\left[1 + G_{b_1b_2}^2 + \frac{8}{3}G_{b_1b_2}\right] + \\ 2\sum_{b_1>b_2>b_3}\left[G_{b_1b_2} + G_{b_2b_3} + G_{b_1b_3} + G_{b_1b_2}G_{b_2b_3} + G_{b_1b_3}G_{b_3b_2} + G_{b_1b_2}G_{b_1b_3}\right] + \\ 4\sum_{b_1>b_2>b_3>b_4}\left[G_{b_1b_2}G_{b_3b_4} + G_{b_1b_4}G_{b_2b_3}\right] = -4\left(1 + G_{l_1l_2}\right)^2 = -\frac{16}{\pi^2} \ . \end{aligned} \tag{59}$$

$Q_2(p)$ is given in (36) and equals zero for any closed path. This can be most easily seen from the cyclic permutations of links under the trace in (55). From (21) follows that

$$Q_3^{(2)}(p) = 0$$

and

$$Q_3^{(3)}(p) = \frac{1}{4} \sum_{b_1 b_2} \sum_{x,x'} \sum_{i<j}^{4} \sum_{i'<j'}^{4} I_2(p) \ . \tag{60}$$

Only one term survives the summation over b_i. Eq.(22) gives

$$Q_3^{(3)}(p) = \sum_{x,x'} \delta_{x,0} \delta_{x',0} \sum_{i<j}^{4} \sum_{i'<j'}^{4} (G_{l_i,l'_{j'}} G_{l_j,l'_{i'}} - G_{l_i,l'_{i'}} G_{l_j,l'_{j'}}) \ . \tag{61}$$

Performing all summations we finally get

$$Q_3^{(3)}(p) = 4\,(1 + G_{l_1 l_2})^2 = \frac{16}{\pi^2} \ . \tag{62}$$

One sees that this connected piece cancels exactly contribution of the second order term from the expansion of the correlation function. We get convinced that

$$\Gamma_p^{(2)} = 0 \ .$$

As a matter of fact, Eqs.(18), (21) and (22) are short-hand statements of the independence of the correlation function of the path. Using these equations one deforms any path expressing some link from a given plaquette through remaining ones. The generalization of this proof to the product of link matrices along an arbitrary closed path is straightforward.

4.1 Infrared finitness of $\Gamma^{(2)}(x,y)$

In this subsection we prove the infrared finitness of $\Gamma^{(2)}(x,y)$. The only term which includes $G_{x,x'}$ and hence could logarithmically diverge is

$$Q_3^{(div)} = \sum_{b_1 b_2} \sum_{x,x'} \sum_{i<j}^{4} \sum_{i'<j'}^{4} G_{x,x'} I_1 \tag{63}$$

and I_1 is given in Eq.(42). We want to show that $G_{x,x'}$ in the last expression can actually be replaced by $G_{x,x'} - G_0$ because all additive constants like logarithmic term in $G_{x,x'}$ do not contribute to the sums in Eq.(63). Thus, one has to prove that

$$S = \sum_{b_1 b_2} \sum_{x,x'} \sum_{i<j}^{4} \sum_{i'<j'}^{4} I_1 = 0 \ . \tag{64}$$

Indeed, using (13) S can be transformed to

$$S = \frac{1}{2} \sum_{b_1 b_2} \sum_{b} \left[A(b_1 b) - A(b b_1)\right] \left[A(b_2 b) - A(b b_2)\right] \ , \tag{65}$$

where sum over b runs over all links of the lattice and

$$A(b_1 b) = \sum_{x} \sum_{i<j}^{4} G_{l_i b_1} G_{l_j b} \ . \tag{66}$$

The equality

$$A(b_1 b) - A(b b_1) = 0 \tag{67}$$

follows from (21) and (23). Therefore, $S = 0$ and Eq.(63) is infrared finite.

5 One-dimensional models in the link formulation

To analyze $1D$ $SU(N)$ model we note that the formula for $\Gamma^{(2)}(x,y)$ given in section 3.2 remains valid if we take for the link Green function the following expression

$$G_{ll'} = 2\delta_{l,l'} \ . \tag{68}$$

This equation is a trivial consequence of the fact that in the link formulation the $1D$ model reduces to one link integrals. Then, it is straightforward to calculate, for example, the first order coefficient of the free energy. We find

$$\frac{1}{L} C^1_{f.e.} = -\frac{N^2-1}{8N} \ , \tag{69}$$

which agrees with the expansion of the exact result in the TL. On the other hand, it is well known that the low-temperature expansion in $1D$ non-abelian models is non-uniform in the volume, in particular the conventional PT produces result different from the Eq.(69). To explain this non-uniformity we remind that on the periodic lattice one has to constraint a holonomy operator if one works in the link formulation. Working with $2D$ models we have neglected this additional constraint since it seems to us rather unlikely that such global constraint may influence the TL in $2D$ (see next section for details). This happens, however in $1D$ model as we are going to show below.

The partition function is given by

$$Z = \int \prod_l dV_l \exp\left[\beta \sum_l \mathrm{Re}\ \mathrm{Tr} V_l + \ln J(V)\right] \ , \tag{70}$$

where $J(V)$ introduces a global constraint on link matrices

$$J(V) = \sum_r d_r \chi_r \left(\prod_{l=1}^{L} V_l \right) \ . \tag{71}$$

Again, at large β we replace the $SU(N)$ delta-function by the Dirac delta-function, i.e.

$$J(V) = \int \prod_{k=1}^{N^2-1} d\phi_k \exp[-i\phi_k \omega^k(C)] \ , \tag{72}$$

where $\omega^k(C)$ is defined as

$$\prod_{l=1}^{L} V_l = \exp[i\lambda^k \omega^k(C)] \ . \tag{73}$$

We omit all technical details which are exactly the same as in $2D$. For the first order coefficient of the free energy we find in the large volume limit

$$\frac{1}{L}C^1_{SU(N)} = -\frac{N^2-1}{8N} + \frac{N(N^2-1)}{24} \ . \tag{74}$$

The second term on the right-hand side of the last formula comes from the expansion of $J(V)$ and modifies the correct expression (69). Our result (74) disagrees with the one given in [2]. We think it is because the result of [2] was obtained using the mass regulator term, i.e. the procedure which is known to give wrong answer even in a finite volume [5]. To check the correctness of (74) we have compared it for $N = 2$ with $O(n = 4)$ model

$$\frac{1}{L}C^1_{O(n)} = \frac{n-1}{8} - \frac{(n-1)(n-2)}{24} \ , \tag{75}$$

where the second term comes from the Hasenfratz term which survives the TL in $1D$. One sees that results indeed coincide[6].

6 Holonomy operators in $2D$

On $2D$ lattice one should restrict two holonomy operators. In our previous analysis we have neglected this restriction. Since, however this global constraint influences the TL in $1D$ if the low-temperature expansion is done in a finite volume we think it is instructive to see what happens with holonomies in two-dimensional models.

Let $H_i (i = 1, 2)$ be any given path winding around the whole lattice. H_1 and H_2 are orthogonal to each other. One has to introduce two global constraints into the partition function (1)

$$J(H) = \int \prod_{k=1}^{N^2-1} \prod_{i=1}^{2} d\phi_k(i) \exp[-i\phi_k(i)\omega^k(H_i)] \ , \tag{76}$$

where $\omega^k(H_i)$ is defined as

$$\prod_{l \in H_i} V_l = \exp[i\lambda^k \omega^k(H_i)] \ . \tag{77}$$

There are two types of contributions from $J(H)$. The first one comes from the expansion of $J(H)$ itself. It is too cumbersome to be given here in full. This contribution, however can be expressed only through link Green functions and is proportional to the linear size of the lattice. This is a reason why it survives the TL in $1D$. Correspondingly, in $2D$ it vanishes like $O(1/L)$. The second type is related to the modification of the generating functional. Namely, one should make the following replacement of the Green functions

$$G_{ll'} \to \bar{G}_{ll'} = G_{ll'} - \frac{1}{2}\sum_i \sum_{(bb') \in H_i} G_{lb}G_{l'b'} \ , \tag{78}$$

[6] One needs also to replace $\beta \to 2\beta$ in $O(4)$.

$$D_l(x) \to \bar{D}_l(x) = D_l(x) - \frac{1}{2}\sum_i \sum_{(bb')\in H_i} D_b(x) G_{lb'} \ , \tag{79}$$

$$G_{x,x'} \to \bar{G}_{x,x'} = G_{x,x'} + \frac{1}{2}\sum_i \sum_{(bb')\in H_i} D_b(x) D_{b'}(x') \ . \tag{80}$$

In particular, the corresponding replacements should be made in formulae (28)-(44). Let us take for simplicity such paths H_i which consist of links pointing only in one direction. The coordinates of the corresponding links on the dual lattice are

$$b = (x_1, 0; n_2) \ x_1 \in [0, L-1] \ , \ b \in H_1$$

and

$$b = (0, x_2; n_1) \ x_2 \in [0, L-1] \ , \ b \in H_2 \ .$$

It is easy to find

$$\sum_{b\in H_1} G_{lb} = 0 \ , \ l = (x, n_1) \ , \tag{81}$$

$$\sum_{b\in H_1} G_{lb} = \frac{2}{L} \ , \ l = (x, n_2) \ . \tag{82}$$

In general we get for $l = (x, n)$, $l' = (x', n')$

$$\frac{1}{2}\sum_i \sum_{(bb')\in H_i} G_{lb} G_{l'b'} = \frac{2}{L^2}\delta_{nn'} \ . \tag{83}$$

On the other hand, one finds in the TL

$$\begin{aligned} \sum_{b\in H_1} D_b(x) &= 2\delta_{x_2,0} - 1 \ , \\ \sum_{b\in H_2} D_b(x) &= 2\delta_{x_1,0} - 1 \ , \end{aligned} \tag{84}$$

which leads to

$$\frac{1}{2}\sum_i \sum_{(bb')\in H_i} D_b(x) D_{b'}(x') = \frac{1}{2}\sum_{n=1}^{2}(2\delta_{x_n,0} - 1)(2\delta_{x'_n,0} - 1) \ . \tag{85}$$

We thus have for new functions

$$\bar{G}_{ll'} = G_{ll'} - O(1/L^2) \ , \tag{86}$$

$$\bar{D}_l(x) = D_l(x) - O(1/L) \ , \tag{87}$$

$$\bar{G}_{x,x'} = G_{x,x'} + \frac{1}{2}\sum_{n=1}^{2}(2\delta_{x_n,0} - 1)(2\delta_{x'_n,0} - 1) \ . \tag{88}$$

One sees that the only new term which could potentially survive the TL is the second term in the last expression. This term is to be substituted into (41) and it leads to the computation of sums of the form

$$P = \frac{1}{8} \sum_{b_1 b_2} \sum_{x,x'} \sum_{i<j}^{4} \sum_{i'<j'}^{4} (4\delta_{x_1,0}\delta_{x_1',0} - 4\delta_{x_1,0} + 1) I_1 \ . \tag{89}$$

The first term vanishes like $O(1/L)$ because of 2 extra deltas. The second and constant terms equal zero because of (64). Moreover, in general it is clear from (86)-(88) that the holonomies may only contribute to the TL through the constant term in $\bar{G}_{x,x'}$. This is, however equivalent to non-cancelation of infrared divergences in some higher orders. In all other cases the holonomies do not survive the TL.

7 Summary and Discussion

In this paper we used the link representation to develop the low-temperature expansion of some $2D$ models. Our main concern here was the investigation of a fixed distance correlation function in models with non-abelian $SU(N)$ global symmetry. We have calculated the first and second order coefficients of large-β expansion of such correlation in terms of Green functions and have shown that they coincide with the conventional expansion, at least for the shortest distance. We have demonstrated how the path independence of the correlation function manifests itself in our expansion and have proven such independence for the first two coefficients via direct calculations. We have also shown which properties of the expansion guarantee its infrared finitness, at least in lowest orders. Moreover, since in our expansion the source of such divergences is exactly localized (they can only emerge from the expectation values of auxiliary fields which appear in even general power) we think it should be possible to generalize the proof given in Section 4.1 for higher orders. It could lead to a lattice analog of David's theorem [6] which states the infrared finitness of the weak coupling expansion of continuum models.

Further, we have re-analyzed the low-temperature expansion of $1D$ models in the link representation. In this representation one should impose a global constraint on link matrices (71). This global constraint vanishes in the TL if this limit is taken before low-temperature expansion. However, if expansion is done in a finite volume, the expansion of the holonomy operator, which imposes the global constraint does survive the TL. It leads to the non-uniformity of the low-temperature expansion in one dimension. We also have proven that it is not the case in $2D$: at least in the lowest orders the holonomies do not survive the TL and there is a good reason to believe that they do not in higher orders as well.

We thus find that the low-temperature expansion performed in the link representation coincides with the conventional PT, at least in the lowest orders. In fact, it seems that these two expansions have to coincide up to arbitrary order. If the conventional PT produces the correct asymptotic expansion in a finite volume, any other expansion is bound to reproduce the same coefficients when the volume is fixed. Moreover,

this also refers to the TL if these coefficients are infrared finite. Originally (following arguments given in [1]) we thought that the expansion around $V_l = I$ has somehow better theoretical status. In particular, if the expansion around $U_x = I$ fails in the TL as the expansion around the non-existing in the TL saddle point, there seems to be a little doubt that the point $V_l = I$ is the only saddle point in the TL similarly to the abelian case. It led us to a conclusion that the expansion in the link representation could be free of ambiguities found in [3] (see also discussion in [7]-[9]) and which put under doubts that the conventional PT gives an asymptotic uniform in the volume for non-abelian models. Unfortunately, this our conclusion was wrong as the next example shows. We consider this example as an superinstanton example [3] adjusted for the periodic lattice in the link formulation. Since the calculations are very lengthy we only sketch the arguments leaving details for a future [10].

Let us consider the system on the periodic lattice given by the partition function (1) with the only difference: we leave one plaquette, say p_0, unrestricted, i.e. on the dual lattice we have ($p_0 \to x_0$)

$$J(V) = \prod_{x \neq x_0} \left[\sum_r d_r \chi_r \left(\prod_{l \in p} V_l \right) \right] . \tag{90}$$

It has to be clear that this modification should leave no memory in the TL. For example, it is the case in the high-temperature expansion (truly, we checked only the lowest order). We are going to argue that while the coefficients of the low-temperature expansion of the XY model are indeed unchanged in the TL, it is not so for non-abelian models and the trouble starts from $1/\beta^2$, precisely like in the superinstanton example. The expansion is done precisely like with the standard partition function, the only modification concerns the generating functional. It can be shown that one has to use the following Green functions

$$G_{ll'} \to M_{ll'} = G_{ll'} + \frac{1}{G_0} D_l(x_0) D_{l'}(x_0) , \tag{91}$$

$$D_l(x) \to M_l(x) = D_l(x) - \frac{1}{G_0} G_{x,x_0} D_l(x_0) , \tag{92}$$

$$G_{x,x'} \to M_{x,x'} = G_{x,x'} - \frac{1}{G_0} G_{x,x_0} G_{x',x_0} . \tag{93}$$

As the simplest examples, we give the expression for the average plaquette in the XY model up to the first order

$$\Gamma_{XY}(p = x) = 1 - \frac{1}{\beta} \frac{\delta_{x,x_0}}{G_0} + O(\beta^{-2}) \tag{94}$$

and the expression for an average link of the unrestricted plaquette up to the second order

$$\Gamma_{XY}(x_0, x_0 + n) = 1 - \frac{1}{4\beta}(1 + \frac{1}{4G_0}) - \frac{1}{32\beta^2}(1 - \frac{1}{16G_0^2}) + O(\beta^{-3}) . \tag{95}$$

One sees that even links of unrestricted plaquette converge to the true TL, though the convergence is slow, like $O(1/\ln L)$ (similarly to the PT with superinstanton BC).

The first order coefficient of the correlation function of the $SU(N)$ model also converges logarithmically to the TL value (29). However, it is not the case for the second order coefficient. This can be most easily seen, e.g. from the expression for the function $M_l(x)$ in Eq.(92). In the case when all sums converge to the TL uniformly, one gets in the TL the following expression

$$M_l(x) = D_l(x) - D_l(x_0) \ , \tag{96}$$

which shows that there appear new contributions which do not vanish in the TL. We have not finished calculations of all these contributions for links of restricted plaquette but it seems for us that even in this case one should expect a modification of the standard TL.

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